SECTION 16.1: DOUBLE INTEGRALS OVER RECTANGULAR REGIONS

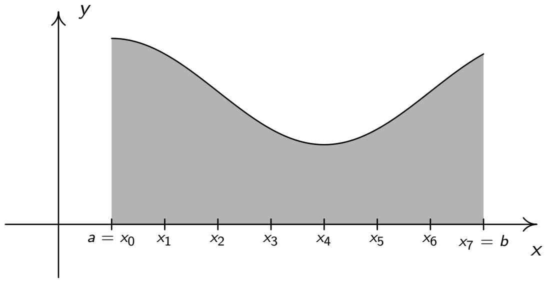
RECALL: The story of the definite integral from Calculus 1 . . .

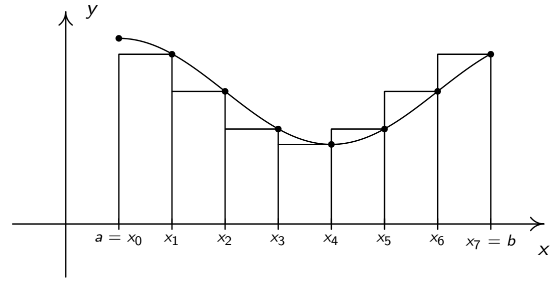
The Area Problem: We wish to find the area between the graph of a continuous function $y = f(x)$ and the x-axis over some closed, finite interval $[a, b]$. We began using rectangle approximations.

To keep things simple, we divided $[a, b]$ into n equal pieces (subintervals), and use the right-endpoints of each piece to determine the height of the rectangles.

We let x_i represent the right endpoint of the ith subinterval, so the height of the ith rectangle is $f(x_i)$. The width of the ith rectangle is the length of the ith subinterval and is denoted $\Delta x_i = x_i - x_{i-1}$.

Below is a depiction of RS_7, a ‘right endpoint sum’ using 7 (equally spaced) subintervals.



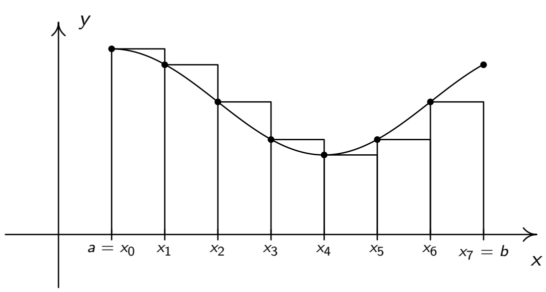


Visualizing RS_7, a ‘right endpoint sum.’

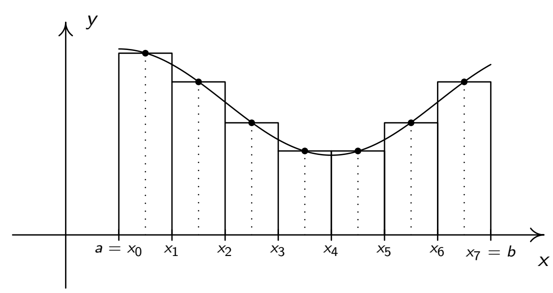
In symbols:

$$\text{Area} \approx f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + ... + f(x_7)\Delta x_7 = \sum_{i=1}^{7} f(x_i)\Delta x_i$$

Instead of using the right endpoint of the subinterval as the location to sample the function, we could also use the left endpoint, or even the midpoint, of the interval, as indicated below:



Visualizing LS_7, a ‘left endpoint sum.’



Visualizing MS_7, a ‘midpoint sum.’

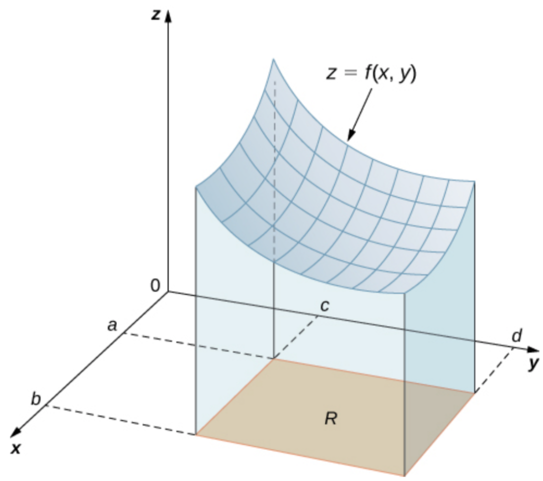
No matter what values we use, we are approximating the area under the curve by the sum of the areas of the rectangles. To get a better approximation of the actual area, we can use more rectangles. Ultimately, we let the number of rectangles n approach infinity and obtained the definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \int_a^b f(x)\,dx$$

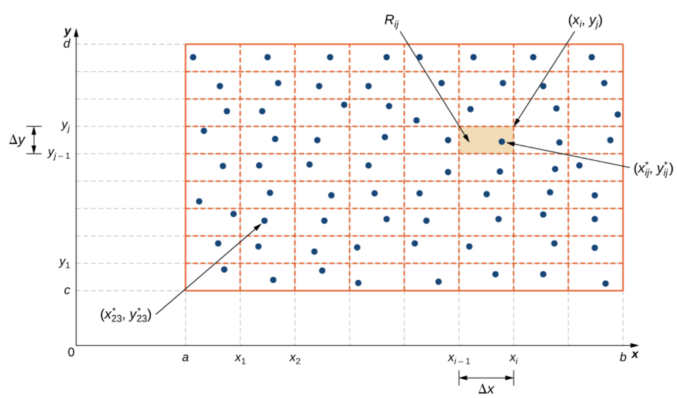
Luckily, we soon developed the **FUNDAMENTAL THEOREM OF CALCULUS** which allowed us to evaluate definite integrals without appealing to the limit process: If F is any antiderivative of f on $[a, b]$, then:

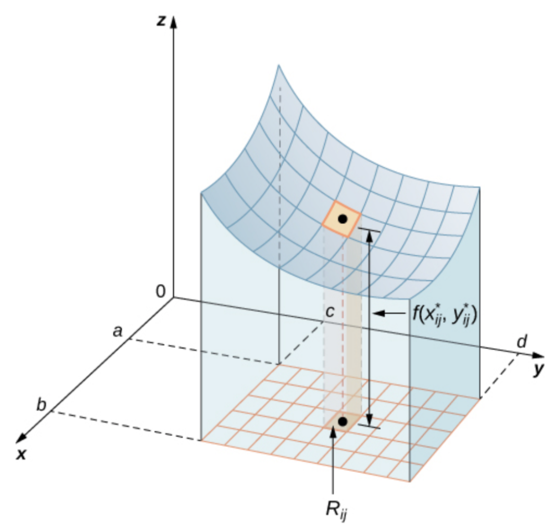
$$\int_a^b f(x)\,dx = F(b) - F(a)$$

The Volume Problem: In Calculus 3, we consider a continuous function f defined over a closed bounded region R and consider the **volume** between the **surface** $z = f(x, y)$ and the xy-plane.



Chopping up the x-axis into m equal pieces and the y-axis into n equal pieces, we divide up the region R into a grid as seen below on the left. We may now approximate the volume under the surface by calculating the volume of **rectangular prisms** whose bases are the rectangles comprising the grid R_{ij} and whose heights are determined by sampling the function at some point, say (x_{ij}^*, y_{ij}^*), within each small rectangle R_{ij}. If we set $\Delta A = \Delta x \, \Delta y$, the area of R_{ij}, then we may approximate the volume of one of these prisms by $V_{ij} = f(x_{ij}^*, y_{ij}^*)\,\Delta A$.





Adding up all the volumes of all of these rectangular prisms as i runs from 1 to n and j runs from 1 to m can be denoted using a **double sum**.

$$\text{Volume} \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

To get a better approximation of the volume, we let $n \to \infty$ and $m \to \infty$ and this produces the **double integral**:

$$\lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = \iint_R f(x, y)\,dA$$

EXAMPLE 1: The values of the function f on the rectangle $R = [0,1] \times [1,4]$ are summarized below.

	$y_0 = 1$	$y_1 = 2$	$y_2 = 3$	$y_3 = 4$
$x_0 = 0$	3.2	1.1	-1.4	-2.5
$x_1 = 0.5$	2.2	-1.3	4.5	5.2
$x_2 = 1$	-1.6	5.3	7.2	9.1

Approximate $\displaystyle\iint_R f(x,y)\,dA$ by using a Riemann Sum with $m = 2$ and $n = 3$.

Select the sample points to be the function values of the upper right corners of the subsquares of R.

Ans: $\displaystyle\iint_R f(x,y)\,dA \approx 15$

PROPERTIES OF THE DOUBLE INTEGRAL Suppose f and g are continuous over R and c is a constant.

- $\displaystyle\iint_R [f(x,y) \pm g(x,y)]\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$
- $\displaystyle\iint_R c\, f(x,y)\, dA = c \iint_R f(x,y)\, dA$
- If $f(x,y) \le g(x,y)$ on R, then $\displaystyle\iint_R f(x,y)\, dA \le \iint_R g(x,y)\, dA$
- If $R = R_1 \cup R_2$ where R_1 and R_2 have an overlap of no area, then $\displaystyle\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA$

QUESTION: How do we evaluate double integrals without resorting to the limit process?

ANSWER: Iterated integrals - that is, using the FToC one variable at a time ...

EXAMPLE 2: Assuming x and y are independent variables, find:

1. $\displaystyle\int_{-1}^{2} 6xy^2\, dy$

   Ans: $\displaystyle\int_{-1}^{2} 6xy^2\, dy = 18x$

2. $\displaystyle\int_{0}^{3} \left[\int_{-1}^{2} 6xy^2\, dy\right] dx$

   Ans: $\displaystyle\int_{0}^{3} \left[\int_{-1}^{2} 6xy^2\, dy\right] dx = 81$

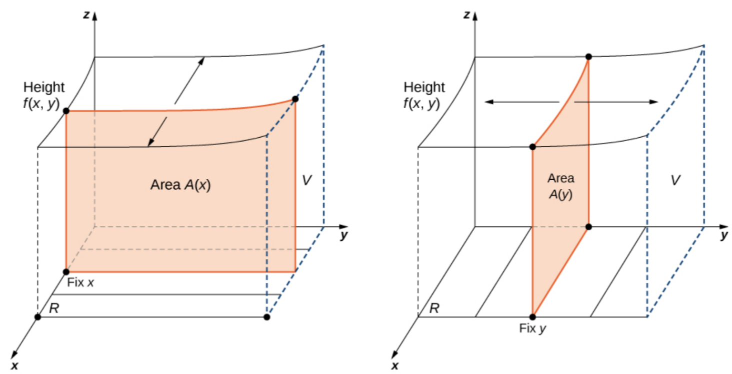
3. $\displaystyle\int_{-1}^{2} \left[\int_{0}^{3} 6xy^2\, dx\right] dy$

   Ans: $\displaystyle\int_{-1}^{2} \left[\int_{0}^{3} 6xy^2\, dx\right] dy = 81$

IN GENERAL: We interpret $\displaystyle\int_a^b \int_c^d f(x,y)\, dy\, dx = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx$

FUBINI'S THEOREM: If f is continuous, and R is the rectangle $\{(x, y) : a \le x \le b,\ c \le y \le d\}$ then:

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$



EXAMPLE 3: Find the volume beneath the surface $z = \dfrac{x}{y}$ above the region $R = \{(x, y) : 0 \le x \le 1,\ 1 \le y \le e\}$.

Ans: Volume $= \displaystyle\int_0^1 \int_1^e \frac{x}{y}\, dy\, dx = \frac{1}{2}$ units^3

EXAMPLE 4: Evaluate $\displaystyle\int_{-1}^{1} \int_0^2 (4 - x - 2y)\, dy\, dx$. Check your answer by switching the order of integration.

Ans: $\displaystyle\int_{-1}^{1} \int_0^2 (4 - x - 2y)\, dy\, dx = 8.$

EXAMPLE 5: Consider the integral: $\int_0^1 \int_1^2 xe^{xy}\, dx\, dy$

1. Attempt to evaluate this integral as written: $\int_0^1 \int_1^2 xe^{xy}\, dx\, dy$. What difficulties do you encounter?

2. Switch the order of integration and evaluate the integral with the new order. What's the lesson here?

Ans: $\int_0^1 \int_1^2 xe^{xy}\, dx\, dy = \int_1^2 \int_0^1 xe^{xy}\, dy\, dx = e^2 - e - 1$

AVERAGE VALUE

RECALL: If f is continuous over $[a, b]$, then the average value of f is $\overline{f} = \frac{1}{b-a} \int_a^b f(x)\, dx$.

DEFINITION: If f is continuous over a closed, bounded region R, then $\overline{f} = \frac{1}{\text{Area of } R} \iint_R f(x, y)\, dA$

EXAMPLE 6: The temperature of a plate at a point (x, y) is given by: $T(x, y) = x \cos\left(\frac{\pi xy}{4}\right)$.

Find the average temperature of the portion of the plate modeled by $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 1\}$

Ans: $\overline{T} = \frac{8}{\pi^2}$

BONUS TRACKS:

1. Find: $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy$

Ans: $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dx\,dy = -\frac{1}{2}$

2. Find: $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx$

Ans: $\displaystyle\int_0^1 \int_0^1 \frac{x-y}{(x+y)^3}\,dy\,dx = \frac{1}{2}$

3. What gives?

HOMEWORK: Section 16.1: 7 - 55 every other odd.